1. When twice a number is increased by 7, the result is 209. Find the number.

2. **(Multiple Choice)** For your answer write the **capital letter** which corresponds to the correct choice.

   If x represents a negative number, how many of the following **must** be positive?

   I. $0.5x$ II. $x-3$ III. $3-x$ IV. $(-2)x$

   A) 0 B) 1 C) 2 D) 3 E) 4

   Note: Be certain to write the correct capital letter as your answer.

3. Two students in Mrs. Smith's class failed algebra. The number of failures was $8\frac{1}{3}\%$ of the total number of students in Mrs. Smith's class. Find the total number of students in Mrs. Smith's class.

4. The price of a gold chain at Store A varies directly as the length of the chain. If a gold chain that is 16 inches long costs $288, find the number of dollars in the cost of a gold chain that is 18 inches long.

5. A group of 4 people are meeting for lunch today in a restaurant. The first person eats lunch there every 8 days; the second person every 9 days; the third person every 10 days; and the fourth person every 12 days. Find the number of days from today when all 4 will first again be eating lunch in this restaurant on the same day.

6. Let $A=\{5,10,15,20,25\}$. Let x be the sum of the 5 remainders when each element of A is divided by 4. Let y be the sum of the 5 remainders when each element of A is divided by 3. Find the value of $(x+y)$.

7. If $2(x-3)=10$, find the value of $\frac{2(x+7)}{x-3}$.

8. Tom had x dollars on the table which he wished to give to various charities. He found that if he gave \$65 to each charity, he would be \$12 short. So, he decided to give \$62 to each of the same charities as before. By so doing, Tom had \$48 left over. Find the value of x.

9. If $a \neq b$, where $a(x-a)-b(x-b)=0$, solve for x in terms of a and b.

10. A certain article is discounted by 15% from its original marked price. Later, the new price is discounted by 10% to a sales price of \$612. Find the number of dollars in the article's original marked price.

11. If $4x = k+198$ and $6y = k-24$, then for all values of x and y, $k = ax+by+c$ where a, b, and c are integers. Find the value of $(2a+3b+c)$.

12. Given the set: $\{5, x, 2y\}$ where x and y represent positive integers. The arithmetic mean (average) of the three members of the set is 6. Find the largest possible value of the product (xy).

13. For all real x, $\Phi(x) = x^2 + x$. If $\Phi(x) = 56$, find the smallest possible value for x.

14. Working at a constant rate, Bob would need 30 hours to build a certain brick wall. Working at a constant rate, Kay would need 21 hours to build the same wall. Due to magical chemistry, if they work together, a total of 8 more bricks will be laid per hour than if working alone. By working together, they took exactly 11 hours, and 40 minutes to build the wall. Find the number of bricks that the wall contained.

15. Let x and y represent integers such that $2x > 14$ and $3y < 12$. Find the smallest possible value of $|y - x|$.

16. Assume $x \neq 0$. Let the three ordered pairs $(x, 10)$, $(5x, 70)$, and $(3x, y)$ represent three distinct points that all lie on the same line. Find the value of y.

17. A painter is planning to paint a row of four houses. He will paint one of the houses completely white, one of the houses completely gray, one of the houses completely blue, and one of the houses completely yellow. In regard only to the respective colors, in how many different ways can the painter paint the four houses?

18. Given $f(x) = 2x - 5$, for what value(s) of a does $f(a) = f(f(a))$?

19. If x and y are integers such that $-101 < x < -89$ and $6 < y < 15$, find the smallest possible value of the product (xy).

20. If $\frac{5+\sqrt{3}}{3-\sqrt{3}} = \frac{a+b\sqrt{c}}{d}$ where a, b, c and d are integers with no common factors and $d > 0$, what is the ordered quadruple (a, b, c, d)?

2013 RA
Algebra I

Name ANSWERS

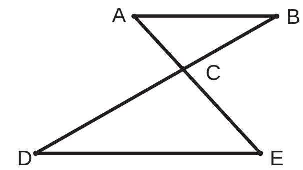
School ______________________________
(Use full school name – no abbreviations)

______ Correct X 2 pts. ea. = [ ]

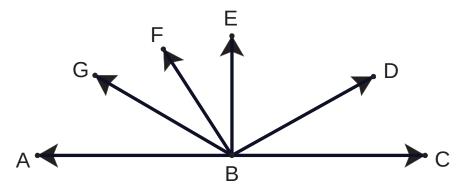
Note: All answers must be written legibly in simplest form, according to the specifications stated in the Contest Manual. Exact answers are to be given unless otherwise specified in the question. No units of measurement are required.

1. 101
2. C (Must be this capital letter.)
3. 24 (Students optional.)
4. 324 ($ or dollars optional.)
5. 360 (Days optional.)
6. 13
7. 6
8. 1288 ($ or dollars optional.)
9. $a+b$ OR $b+a$ (Grouping symbols optional.)
10. 800 ($ or dollars optional.)
11. –74
12. 21
13. –8
14. 1680 (Bricks optional.)
15. 5
16. 40
17. 24 (Ways or arrangements optional.)
18. 5
19. –1400
20. $(9,4,3,3)$ (Must be this ordered quadruple.)

1. In the given diagram, $\overline{AE} \cap \overline{BD} = C$, $\overline{AB} \parallel \overline{DE}$, $CD = 24$ and $BD = 33$. Find the ratio of the perimeter of ΔABC to the perimeter of ΔCDE. Give your answer in the form $a:b$ where a and b are positive integers with no common factors.

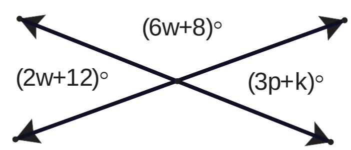


2. In the diagram, $\overleftrightarrow{EB} \perp \overleftrightarrow{AC}$ and points A, B, and C are collinear. $\overrightarrow{BF}$ bisects $\angle GBE$. If $\angle EBF = 24°$ and $\angle CBD = 46°$, find the degree measure of $\angle DBG$.

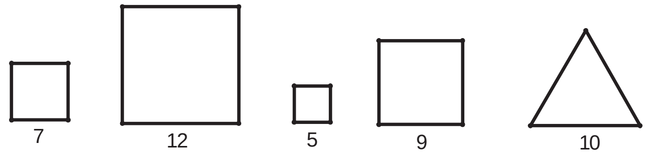


3. If the smaller base of the trapezoid is increased by 12 and the larger base of the trapezoid is decreased by 4, by how much is the median increased or decreased? Give an increase as a positive number and a decrease as a negative number.

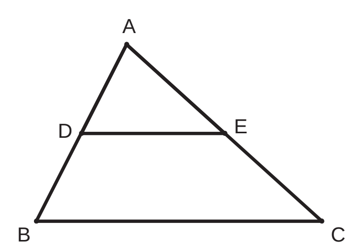
4. The diagram shows two intersecting lines with degree measures as indicated. If $p = 14$, find the value of k.



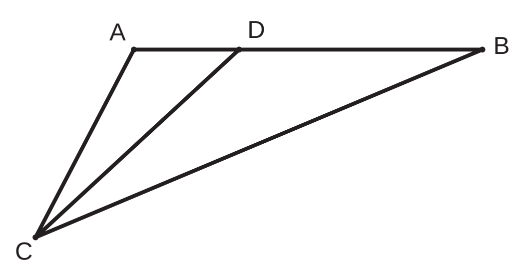
5. The diagrams show 4 squares and one equilateral triangle with side-lengths as indicated. If two distinct polygons of the five are selected at random, find the probability that both polygons selected have a perimeter greater than 25. Express your answer as a common fraction reduced to lowest terms.



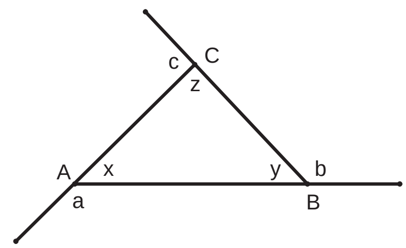
6. In the diagram, not necessarily drawn to scale, points, A, D, and B are collinear, and points A, E, and C are collinear. $\overline{DE} \parallel \overline{BC}$. $AD = 12$, $AE = 18$, and $DB = 4$. Find EC.



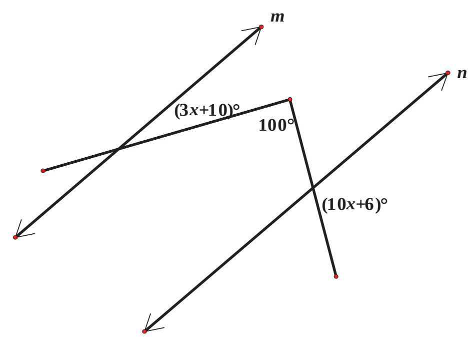
7. In the diagram, points A, D, and B are collinear. $\angle BAC = 112°$, $\angle ACD = 18°$, and $\angle ABC = 34°$. Find the degree measure of $\angle DCB$.



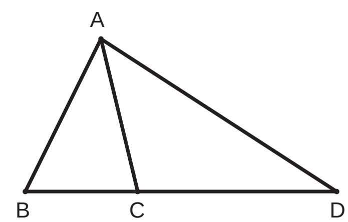
8. In the diagram (not necessarily drawn to scale), x, y, and z are the respective degree measures of the interior angles of ΔABC, and a, b, and c are the respective degree measures of the exterior angles of ΔABC as shown. If $x:y:z=101:473:397$, find $a:b:c$. Express your answer in the form of relatively prime positive integers.



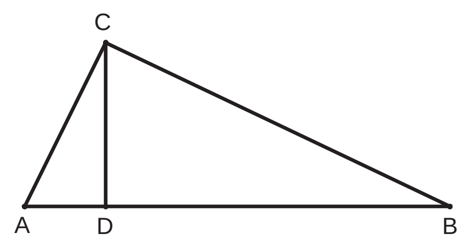
9. Given $m \parallel n$ in the diagram shown, find the value of x. (Note: The three angle measures shown in the diagram are $(3x+10)°$, $100°$, and $(10x+6)°$.)



10. In the diagram (not necessarily drawn to scale), C lies on $\overline{BD}$, and $\angle BAC \cong \angle DAC$. If $AB=7x$, $AD=10x-8$, $BC=5x$, and $CD=6x$, find the value of x.



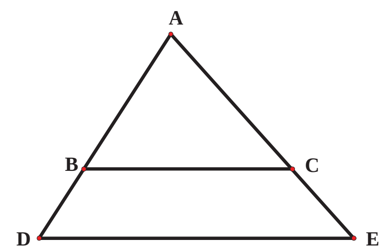
11. In the diagram, points A, D, and B are collinear. $\overline{AC} \perp \overline{CB}$, and $\overline{CD} \perp \overline{AB}$. If $AD=12$ and $DB=108$, find the length of $\overline{CD}$.



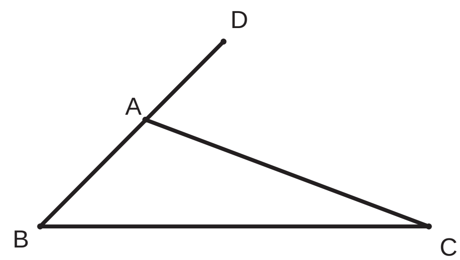
12. The ratio of the area of an equilateral triangle to the area of a square is $25\sqrt{3}:64$. What is the ratio of the perimeter of the equilateral triangle to the perimeter of the square? Give your answer in the form $a:b$ where a and b are integers with no common factors.

13. On a flat surface, Zig starts by going 3 miles due north, then goes 6 miles due west, then goes 2 miles due north, and finally goes 3 miles due west. At the end of his journey, how many miles is Zig from his starting point?

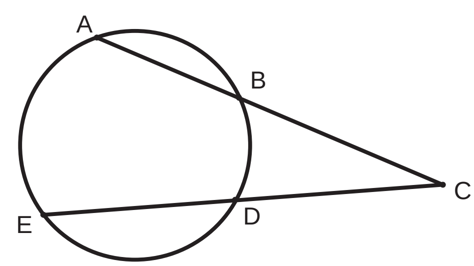
14. Given A, B, D and A, C, E are sets of collinear points. $\overline{BC} \parallel \overline{DE}$, $AB = x - 1$, $BC = 18$, $AC = 2x + 3$, $BD = y$, $CE = 3y + 1$ and $DE = 24$. Find the ordered pair (x, y).



15. In the diagram with ΔABC, points B, A, and D are collinear. If the degree measures of $\angle ABC$, $\angle ACB$, and $\angle BAC$ respectively are $2x + 6$, $x + 1$, and $3x + 11$, find the degree measure of $\angle DAC$.



16. In the diagram, points A, B, D, and E lie on the circle. Points A, B, and C are collinear, and points C, D, and E are collinear. If $CD = 4.6$, $DE = 4.8$, and $BC = 3.2$, find the length of $\overline{AC}$. Express your answer as an **exact decimal**.



17. Let points A, B, C, and D lie on a circle in some order such that $\angle CBD = 85°$, $\angle ADC = 35°$, $\overline{BD}$ is not a diameter, and minor arc $\overset{\frown}{AB} = 60°$. Find the degree measure of **minor** arc $\overset{\frown}{BD}$.

18. A right triangle has legs with lengths $10\sqrt{3}$ and $24\sqrt{3}$. Find the exact area of a circle that is inscribed in this triangle?

19. One of the interior angles of a convex octagon measures $20°$. What is the maximum number of interior angles of that convex octagon that could be supplementary to the $20°$ angle?

20. Circle A has a center $(6, -2)$ and radius 5. Circle B has center $(-3, 4)$ and radius 3. Find the exact length of the common internal tangent between circles A and B?

2013 RA
Geometry

Name ANSWERS

School ______________________________
(Use full school name – no abbreviations)

______ **Correct X 2 pts. ea. =** [ ]

Note: All answers must be written legibly in simplest form, according to the specifications stated in the Contest Manual. Exact answers are to be given unless otherwise specified in the question. No units of measurement are required.

1. $3:8$ (Must be this ratio in this form.)
2. 92 (Degrees optional.)
3. 4
4. 10
5. $\frac{3}{5}$ (Must be this reduced common fraction.)
6. 6
7. 16 (Degrees optional.)
8. $435:249:287$ (Must be this ratio in this form.)
9. 12
10. 5
11. 36
12. $15:16$ (Must be this ratio in this form.)
13. $\sqrt{106}$ (Must be this exact answer, miles optional.)
14. $\left(3, \frac{2}{3}\right)$ OR $\left(3, 0.\overline{6}\right)$ (Must be this ordered pair with exact coordinates.)
15. 88 (Degrees optional.)
16. 13.5125 (Must be this decimal.)
17. 60 (Degrees optional.)
18. 48π (Must be this exact answer.)
19. 6 (Angles optional.)
20. $\sqrt{53}$ (Must be this exact answer.)

1. If $\log_2(k) = 9$, solve for k.

2. Given the functions defined by $f = \{(-3,-8),(2,7),(5,3),(7,10)\}$ and $g(x) = x^2 + 1$. Find the value of x for which $f(x) = g(-3)$.

3. When $(x^5 - 2x^3 + x + k)$ is divided by $(x-2)$, the remainder is 2. Find the value of k.

4. If the sum of the consecutive **odd** integers from -11 to n inclusive (including -11 and n) is 189, find the value of n.

5. Let x represent a positive integer and let y represent a real number. Find the **sum** of all distinct possible values for x if $y = \sqrt{10-x}$.

6. Find the value of k if the graph of $f(x) = \dfrac{3kx - 10}{4x + k}$ has an x-intercept of 30. Express your answer as a reduced common or improper fraction.

7. How many distinct ordered pairs (x, y) of positive integers exist such that $x + y \leq 12$?

8. If $\begin{bmatrix} x & y \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}+\begin{bmatrix} 2 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 13 & 8 \end{bmatrix}$, find the value of $(x+y)$. Express your answer as a common or improper fraction reduced to lowest terms.

9. Assume that the probability that D. Rose will make any free throw is 73%. Find the maximum whole number of free throws that D. Rose can shoot such that his probability of making at least one free throw is less than 99%.

10. Let $f(x) = 27x^2 - 32x + 10$. Find the absolute value of the difference between the distinct values of x such that $f(x) = x$. Express your answer as a reduced common or improper fraction.

11. The first term of a geometric sequence is $\frac{2}{3}$ and the second term is $\frac{1}{3}$. Find the sum of the fourth and fifth terms of this geometric sequence. Express your answer as a reduced common or improper fraction.

12. Right triangle ΔABC with $\angle ACB = 90°$ lies in plane m with $AC = 5$ and $BC = 12$. Points D, E, and F lie on the same side of the plane such that $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ are each perpendicular to plane m. If $AD = 18$, $BE = 20$, and $CF = 27$, find the perimeter of ΔDEF.

13. Find the length of the minor axis of the conic section whose equation is $x^2 - 2x + 3y^2 - 12y + 4 = 0$.

14. Let n be a positive integer such that $1+3+6+\cdots+\frac{n(n+1)}{2} = 23(1+2+3+\cdots+n)$. Find the value of n.

15. Let $i = \sqrt{-1}$. Find the value of $i - i^2 + i^3 + i^4 - i^5 + i^6 - i^7 + i^8$.

16. Let $C(x, y)$ be a symbol for combinations of x distinct things taken y at a time. If $C(k,3) - C(w,3) = 166$, find the least value of $(k + w)$.

17. It is given that $f(x) = \dfrac{1-2x}{3}$ and $g(x) = \dfrac{1-3x}{2}$. Find $f\left(g(7)\right) + g\left(f(8)\right) + f\left(g(9)\right)$.

18. Three monkeys randomly toss 12 ping pong balls numbered consecutively from 1 through 12, inclusive, into three containers labeled A, B, C. If each ball goes in one of the containers and is equally likely to end up in any one of the containers, find the number of distinct outcomes that are possible in which all containers have the same number of balls. Express your answer as an integer. The order in which the balls are placed in the containers does not matter.

19. Find the value of k for which the graph of $y = 16x^2 + 2x + k$ is tangent to the x-axis. Express your answer as a reduced common or improper fraction.

20. Let a and b be real numbers. Let a be an even integer greater than -53 and less than 59. The largest possible value of the sum of the squares of the roots for x in the cubic equation $x^3 - ax^2 + ax\sqrt{74} + b = 0$ can be written in simplest radical form as $k + w\sqrt{p}$. Find the value of $(k + w + p)$.

2013 RA

Algebra II

Name ANSWERS

School ______________________

(Use full school name – no abbreviations)

______ **Correct X 2 pts. ea. =** [ ]

Note: All answers must be written legibly in simplest form, according to the specifications stated in the Contest Manual. Exact answers are to be given unless otherwise specified in the question. No units of measurement are required.

1. 512

2. 7

3. –16

4. 29

5. 55

6. $\frac{1}{9}$ (Must be this reduced common fraction.)

7. 66 (Ordered pairs optional.

8. $\frac{7}{6}$ (Must be this reduced improper fraction.)

9. 3 (Must be this whole number.)

10. $\frac{1}{9}$ (Must be this reduced common fraction.)

11. $\frac{1}{8}$ (Must be this reduced common fraction.)

12. $\sqrt{106}+\sqrt{173}+\sqrt{193}$ (Terms can be in any order, must be exact.)

13. $2\sqrt{3}$ (Must be this exact answer.)

14. 67

15. 2

16. 23

17. 24

18. 207900 (Must be this integer, outcomes optional.)

19. $\frac{1}{16}$ (Must be this reduced common fraction.)

20. 2882 (Must be this integer.)

1. The tangent of one of the acute angles of a right triangle is 1. If one of the legs of this right triangle has a length of $7\sqrt{2}$, find the length of the hypotenuse of this right triangle.

2. If the indicated inner product (dot product) of the vectors, that is $(5,6)\cdot(-3,k)$, is 27, find the value of k.

3. The first term of a geometric sequence is 1, the second term is -2, and the third term is 4. Find the **sum** of the fifth and seventh term.

4. The base of an equilateral triangle lies on the *x*-axis. Find the largest slope of any of the three lines containing the sides of this equilateral triangle.

5. Find the length of the minor axis of the conic section whose equation is $5x^2 - 10x + 9y^2 - 72y + 104 = 0$.

6. Let $\left\|(k,w)\right\|$ represent the **norm** of the vector represented by (k,w). Find the value of $\left\|\left(\frac{3}{5},\frac{4}{5}\right)\right\|$.

7. Find the sum of all distinct positive values of y that satisfy the system: $\begin{cases}\log_4(x)=\log_2(y)\\ x^2-9y^2+8=0\end{cases}$.

8. The sum of the first four terms of an arithmetic sequence is 106, and the sum of the first five terms of this arithmetic sequence is 180. Find the **second** term of this sequence.

9. Let k, w, and p represent positive integers such that two of the roots for x of the cubic equation $x^3 + kx^2 - wx - p = 0$ are additive inverses. If p is the smallest positive number that has exactly 8 distinct positive integral divisors, find the smallest possible value of $(k+w)$.

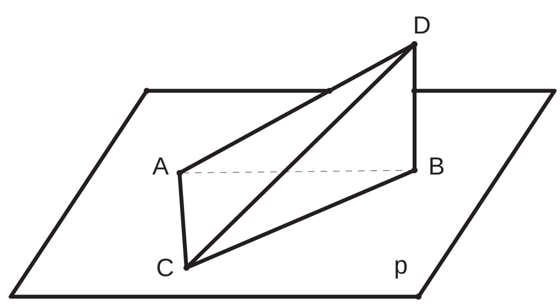
10. If -7 and $\frac{1}{2}$ are two of the roots for x in the equation $6x^3 + kx^2 + 44x + w = 0$, find the value of $(k+w)$.

11. An ellipse has an equation of $6x^2 + y^2 = 24$. One of the foci of this ellipse is located at $(0, k\sqrt{w})$ where k and w are positive integers. Find the smallest possible value of $(k+w)$.

12. Two numbers are selected at random in the interval $[0,5]$. Find the probability that the sum of the squares of these numbers is less than 5. Express your answer as a **decimal** rounded to the nearest **ten-thousandth**.

13. Let $i = \sqrt{-1}$. If $|28 + ki| = 53$ and $k < 0$, find the value of k.

14. In the diagram, not necessarily drawn to scale, A, B, and C lie in plane p. $\angle CDA = 15^\circ$, and $\overline{DB} \perp p$. $AC = 6$, $BC = 10$, and $AB = 8$. In reduced radical form, $AD = k + w\sqrt{p}$. Find the sum $(k+w+p)$.



15. Let f be a real-numbered function. If $f(x)=\dfrac{\log(x-2)}{x^2-40}+\sqrt{8\sqrt{3}-x}$, then the domain of f, given in simplified radical form in interval notation, is $(k, a\sqrt{b})\cup(c\sqrt{d}, g\sqrt{h}]$ where k, a, b, c, d, g, and h are positive integers. Find the value of $3k+2a+b+c+d+7g+h$.

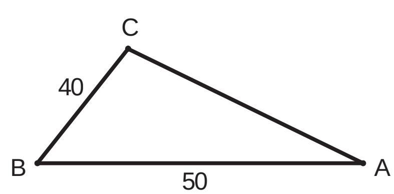
16. The vector $(7.1,\ 8.2)$ is perpendicular to the vector $(-4.1,\ k)$. Find the value of k. Express your answer as an **exact decimal**.

17. If the eight letters of the word *KANKAKEE* are written in random order, find the probability that the three $K's$ are consecutive letters. Express your answer as a common fraction reduced to lowest terms.

18. Find the sum of all (not necessarily distinct) y-coordinates for point(s) of intersection of the graphs of $x^2+y^2=16$ and $x^2=15y$.

19. The sum of the infinite geometric series whose first term is 900 is 2250. Find the second term of this infinite geometric series.

20. In the diagram, $AB=50$ and $BC=40$. If $\dfrac{\sin\left(0.5\left(\angle CAB-\angle CBA\right)\right)}{\cos\left(0.5\left(\angle ACB\right)\right)} = 0.48$, find AC.



2013 RA

Pre-Calculus

Name ANSWERS

School ______________________________

(Use full school name – no abbreviations)

______ Correct X 2 pts. ea. = [ ]

Note: All answers must be written legibly in simplest form, according to the specifications stated in the Contest Manual. Exact answers are to be given unless otherwise specified in the question. No units of measurement are required.

1. 14
2. 7
3. 80
4. $\sqrt{3}$ (Must be this answer.)
5. $2\sqrt{5}$ (Must be this exact answer.)
6. 1
7. $1+2\sqrt{2}$ OR $2\sqrt{2}+1$ (Must be this exact answer.)
8. 17
9. 10
10. 14
11. 7
12. 0.1571 OR .1571 (Must be this decimal.)
13. –45
14. 21
15. 91
16. 3.55 (Must be this decimal.)
17. $\frac{3}{28}$ (Must be this reduced common fraction.)
18. 2
19. 540
20. 16

NO CALCULATORS

1. **(Yes or No)** If $x+y=8$, and $y>15$, does x represent a negative number?

   Note: Be certain to write the whole word "Yes" or "No" for your answer.

2. From the set $\{3, 5, 7, 9, 11, 13, 15, 17\}$, one number is selected at random. Find the probability that the number selected is a prime number. Express your answer as a common fraction reduced to lowest terms.

3. Let x and y represent integers such that $x=2y$. If $(x+14)(3y-13)<0$, find the sum of all possible distinct values of x.

4. **(Multiple** Choice**)** For your answer write the **capital letter** which corresponds to the correct choice.

   If $\dfrac{3k+w}{k-2w}=\dfrac{x}{2w-k}$, then the value of x in terms of w and k is:

   A) $3k+w$ B) $-3k-w$ C) $k+2w$ D) $k-2w$ E) $-k-2w$

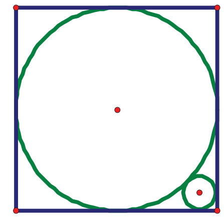
   Note: Be certain to write the correct capital letter as your answer.

5. Bob and Judy were playing the game of "Double." If a person is successful on his/her first turn, thereafter for each successful turn, the number of points added to the person's score is double the person's total score. Judy went first and was successful. She was also successful after each of her next five turns. After adding the points for her sixth successful turn, Judy's score was 3159. How many points did Judy score for her first successful turn?

6. Find the *y*-intercept of the line that is the perpendicular bisector of the line segment joining $(-1, 5)$ and $(3, 2)$. Express your answer as an improper fraction reduced to lowest terms.
   Note: The *y*-intercept is only a value for *y* and is **not** an ordered pair.

7. If each interior angle of a regular polygon is $165°$, how many sides does the polygon have?

8. The larger of the two circles in the diagram is inscribed in a square. The smaller of the two circles is tangent to two sides of the square and the larger circle. If the perimeter of the square is 40 units, find the circumference of the smaller circle.



NO CALCULATORS

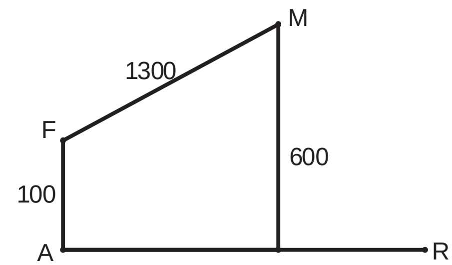
NO CALCULATORS

9. The length of a diagonal of a square is $4.2\sqrt{8}$. Find the perimeter of the square. Give your answer as a **decimal**.

10. In the table shown, use each of the integers 1, 2, 3, 4, 5, 6, 7 exactly once in row 2 and use the same ordered seven-tuple in row 3 as in row 2 so that the sum of the numbers in each column will be the square of an integer. For your answer, give the **ordered seven-tuple** of the form (a,b,c,d,e,f,g)

1	3	5	7	2	4	6
a	*b*	*c*	*d*	*e*	*f*	*g*
a	*b*	*c*	*d*	*e*	*f*	*g*

11. In the diagram (not drawn to scale), Farmer Brown's farmhouse is at point F and is 100 feet due north of the river that runs due west to east as shown from A to R. The farmhouse is 1300 feet from his machine shed M, and the machine shed is 600 feet due north of the river. Farmer Brown needs to install a pump on the river so he can run piping to both the farmhouse and machine shed. Find the number of feet downstream from A toward R that represents the point at which Farmer Brown should put his pump into the river so that his total length of piping from F to the river and M to the river is a minimum. Express your answer as an improper fraction reduced to lowest terms.



12. Find the length of a **diameter** of a circle whose equation is $x^2 - 14x + y^2 + 2y = 71$.

13. Find the product of the slope and the *y*-intercept of the line whose equation is $4x - 3y = -9$. **Note:** the *y*-intercept is the *y*-coordinate **only** of the point at which the line intersects the *y*-axis.

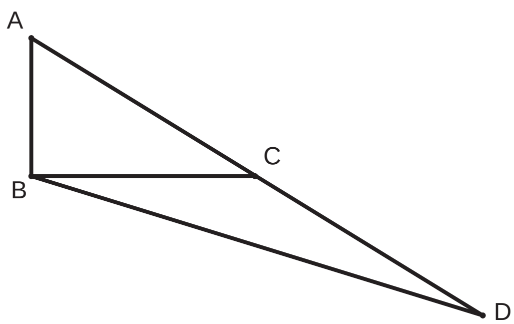
14. Point A is the point $(8,0)$ and point B is the point $(0,b)$ where $b > 0$. Point O is the origin and the area of triangle AOB is 40. If the point $(4,k)$ lies on line $\overleftrightarrow{AB}$, what is the value of k?

NO CALCULATORS

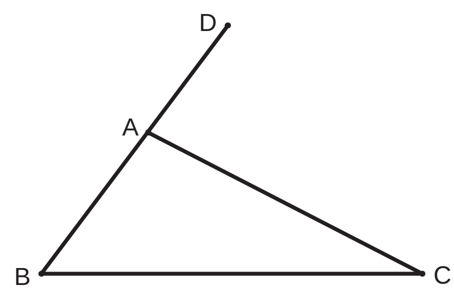
NO CALCULATORS

15. Ida is the inspector for a company. During one of Ida's inspection shifts, she inspected 24 out of 160 items. The ratio of inspected items to uninspected items for this shift can be expressed as $k:w$ where k and w are positive integers. Find the smallest possible value of w.

16. In the diagram, points A, C, and D are collinear. $AB = 6$, $BC = 8$, $\angle CBD = 30°$, and $\overline{AB} \perp \overline{CB}$. Then $CD = \frac{k\sqrt{w}+f}{p}$ where k, w, f, and p are positive integers. Find the smallest possible value of $(k+w+f+p)$.



17. In the diagram, points B, A, and D are collinear. If $\angle DAC = (5x-3)°$, $\angle ABC = (4x-8)°$, and $\angle ACB = (2x-10)°$, find the degree measure of $\angle BAC$.



18. Set A contains 11 elements, set B contains 18 elements, and $A \cap B$ contains 8 elements. How many elements are in $A \cup B$?

19. The roots for y for the equation $y^3 + ay^2 + by + c = 0$ are each two less than the corresponding roots for x for the equation $x^3 - 5x^2 - 8x + 12 = 0$. Find the value of $(a+b+c)$.

20. Solve for y: $\frac{3}{x} + \frac{8}{y} = 10$, $\frac{5}{x} - \frac{4}{y} = 8$

NO CALCULATORS

2013 RA

Fr/So 8 Person

School ANSWERS

(Use full school name – no abbreviations)

_______ Correct X 5 pts. ea. = [ ]

Note: All answers must be written legibly in simplest form, according to the specifications stated in the Contest Manual. Exact answers are to be given unless otherwise specified in the question. No units of measurement are required.

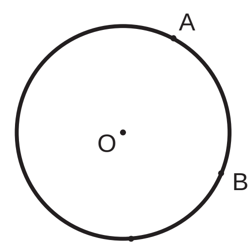
1. Yes (Must be this whole word.)
2. $\frac{3}{4}$ (Must be this reduced common fraction.)
3. –22
4. B (Must be this capital letter.)
5. 13
6. $\frac{13}{6}$ (Must be this reduced improper fraction.)
7. 24
8. $\left(5\sqrt{2}-5\right)\pi$ OR $5\pi\left(\sqrt{2}-1\right)$ (Or simplified equivalent, common factors allowed.)
9. 33.6 (Must be this decimal.)
10. $\left(4,3,2,1,7,6,5\right)$ (Must be this ordered seven-tuple.)
11. $\frac{1200}{7}$ (Must be this reduced improper fraction, feet optional.)
12. 22
13. 4
14. 5
15. 17
16. 294
17. 108 (Degrees optional.)
18. 21
19. –31
20. 2

NO CALCULATORS

1. Name the quadrant in which $\sin(x) < 0$ and $\cos(x) > 0$. Express your answer as a **Roman numeral**.

2. Let $C(n,k) = \dfrac{n!}{k!(n-k)!}$. Find the value of $C(9,3)$.

3. In the diagram, points A and B lie on a circle with center at O. If $\overset{\frown}{AB} = 96°$ and the circumference of the circle is 50π, then the length of $\overset{\frown}{AB}$ is $k\pi$. Find the value of k. Express your answer as a reduced common or improper fraction.



4. For what **negative** value of x will 3, x, and 10, taken in that order, form a geometric sequence?

5. Reading from left to right, the first term in a row of Pascal's triangle is 1 and the second term is 9. Find the sum of the ten terms in this row of Pascal's triangle.

6. **(Always, Sometimes, or Never)** For your answer, write the whole word **Always, Sometimes,** or **Never**—whichever is correct.

   If k represents a positive integer, then $1+3+5+\ldots+(2k-1) = k^2 - 1$.

7. Assuming $x \neq 0$, when $\left(x+\dfrac{1}{x}\right)^4\left(x-\dfrac{1}{x}\right)^4$ is expanded and completely simplified, find the value of the constant term.

8. Find the smallest possible positive value of k such that $(103)(108)(113)(118)+k$ is the square of an integer.

NO CALCULATORS

NO CALCULATORS

9. **(**Always**, Sometimes, or Never)** For your answer, write the whole word **Always, Sometimes,** or **Never**—whichever is correct.

   If k represents a rational number, then k can be represented as a repeating decimal.

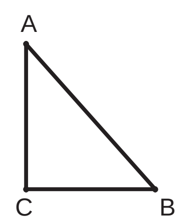
10. The smallest interior angle of a convex polygon with less than 12 sides has a degree measure of 120. Moving clockwise around the convex polygon from the smallest interior angle, each interior angle is 5° greater than the interior angle that precedes it until arriving back at the 120° angle. Find the sum of the measures of the interior angles of this convex polygon.

11. Let N be the number of noncongruent triangles with **all** of the following properties:

    (1) The length of each of the three sides is less than or equal to 8.
    (2) The lengths of at least two sides are integers.
    (3) At least one angle of the triangle is $30°$ or $90°$.
    (4) The area of the triangle is an integer.

    Find the value of N .

12. In ΔABC with right angle at C, $\frac{AC}{BA}=\frac{45}{53}$. Find $\sin\angle BAC$. Express your answer as a common fraction reduced to lowest terms.



13. The sum of the first and fourth terms of an arithmetic progression is 70. If the third term of this arithmetic progression is subtracted from the tenth term of the progression, the result is 84. Find the eighth term of this arithmetic progression.

14. Let a, b, c, and d represent positive integers and let the arithmetic mean of a, b, c, d and e be e. Let k be the smallest possible positive integer such that e **must** be a positive integer if $a+b+c+d$ is an integral multiple of k. Find the value of k.

NO CALCULATORS

NO CALCULATORS

15. Let $i = \sqrt{-1}$. Which quadrant contains the fewest 15th roots of $(-1+i)$? Express your answer as a **Roman numeral**.

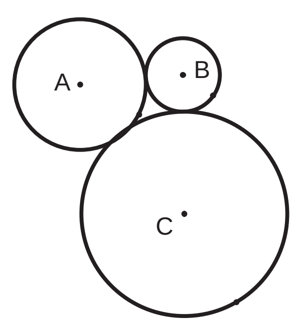
16. Find an integer k such that the equation $x^3 + 19x^2 - 13x + k = 0$ has two roots that are additive inverses of each other.

17. Find the focus of the parabola whose equation is $y = x^2 + 6x + 9.75$. Express your answer as an ordered pair of the form (x, y).

18. Find the **ordered pair** that represents the additive inverse of the vector $(-9, 4)$.

19. Three cubical dice (with faces numbered 1, 2, 3, 4, 5, 6) are randomly tossed at the same time. Two of the dice are fair. The third is unfair with probabilities of the number turned up as follows: $P(1) = \frac{1}{12}$, $P(2) = \frac{1}{3}$, $P(3) = \frac{1}{4}$, $P(4) = \frac{1}{6}$, $P(5) = \frac{1}{12}$, and $P(6) = \frac{1}{12}$. Find the probability that the numbers turned up from the random toss of the three cubical dice can be arranged to form an arithmetic progression for which 2 and 3 are two members of the arithmetic progression. Express your answer as a common fraction reduced to lowest terms.

20. In the diagram, (not necessarily drawn to scale), each circle is tangent to the other two circles. The centers of the circles are A, B, and C. $BC = 10$, and the lengths of the radii (in some order) of circles A and B are in the ratio of $1:4$. If the lengths of the radii of the three circles are all integers, and if the area of ΔABC is an integer, find the largest possible area of ΔABC.



NO CALCULATORS

2013 RA

School ANSWERS

Jr/Sr 8 Person

(Use full school name – no abbreviations)

______ Correct X 5 pts. ea. = [ ]

Note: All answers must be written legibly in simplest form, according to the specifications stated in the Contest Manual. Exact answers are to be given unless otherwise specified in the question. No units of measurement are required.

1. IV (Must be this Roman numeral.)
2. 84
3. $\frac{40}{3}$ (Must be this reduced improper fraction.)
4. $-\sqrt{30}$ (Must be this exact answer.)
5. 512
6. Never (Must be the whole word.)
7. 6
8. 625
9. Always (Must be the whole word.)
10. 1260 (Degrees optional.)
11. 33 (Triangles optional.)
12. $\frac{28}{53}$ (Must be this reduced common fraction.)
13. 101
14. 4
15. III (Must be this Roman numeral.)
16. -247
17. $(-3,1)$ (Must be this ordered pair.)
18. $(9,-4)$ (Must be this ordered pair.)
19. $\frac{17}{216}$ (Must be this rediced common fraction.)
20. 48

Round answers to four significant digits and write in either standard or scientific notation unless otherwise specified in the question. Except where noted, angles are in radians. No units of measurement are required.

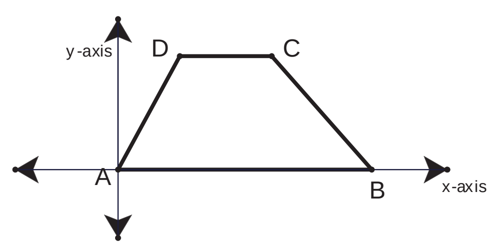
1. Find the value of $\cot(72136°)$.

2. The length of the apothem of a regular polygon containing 22 sides is 5.684. Find the perimeter of this regular polygon.

3. Assume the price of a gallon of gasoline is 425.9 **cents**. If Dick drove 19487 miles, and his car averaged 26.72 miles per gallon of gasoline, find the number of **dollars** that Dick spent for gasoline in driving those 19487 miles. Report your answer as an **integer** rounded to the nearest whole **dollar**.

4. Find the circumference of the circle whose equation is $x^2 + 84.26x + y^2 - 46.88y + 21.64 = 0$.

5. In the diagram, $\overleftrightarrow{AB} \parallel \overleftrightarrow{DC}$, $AD = 8.467$, $DC = 12.03$, $BC = 13.83$ with $A(0,0)$ and $B(23.16,0)$. Find the ***x*-coordinate only** of point C.



6. If $2.239x = 3.447y = 7.612z$, then $4.143z = kx$. Find the value of k.

7. Eagle-eyed Ellen is at the top of a 100 foot vertical tower and spots an ant on the flat horizontal ground at an angle of depression of 20°. Find the number of feet the ant is from the foot of the tower.

8. Team C is playing Team D in a series of a maximum of 7 games. The series ends as soon as one of the two teams has won 4 games. Assume that the probability that Team D will win any game is k and that that probability is constant. Assume also that the probability that Team D will win the series is 0.6947. Find the value of k.

9. Find the simplified value of $\dfrac{(8.450X10^{-3})(2.710X10^{11})}{(4.230X10^{7})(2.950X10^{-13})}$. Express your answer in **scientific notation**.

10. \$1000 is invested in an account at 5% annual percentage rate and is compounded monthly. A second \$1000 is invested in an account at 5% annual percentage rate and is compounded continuously. At the end of 10 years, find the absolute value of the difference between the values of the two accounts. Round the value of each account to the nearest cent at the end of the ten years. Express your answer in **cents** as a **whole number of cents.**

11. Find the value of $\left(18^{0.73}\right)\left(18^{0.09}\right)^{3}\left(18^{-0.5}\right)$.

12. A regular polygon that has 17 sides has a perimeter of 896.9. Find the length of an apothem of the regular polygon.

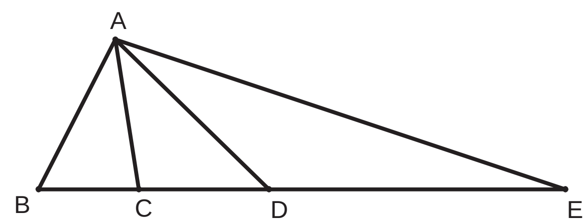
13. A cube is circumscribed by a sphere with radius of length 10.45. Find the volume that is inside the sphere but outside the cube. Express your answer as a **decimal** rounded to the nearest thousandth. **Do not** use scientific notation.

14. If $x = 1.436$, find the value of $(x^{x})^{(x^{(2x)})}$.

15. If $y = 12x - 5x^{3} + 3$, find the minimum value for y when $x < 0$.

16. Ordering online from the Sellalot Company, Bill bought 50 items at $9.43 each, 2 items at $225.63 each, and 152 items at $10.99 each. Ordering online from the Sellalot Company, Becky bought 30 items at $13.45 each, 3 items at $99.99 each, and 4 items at $19.99 each. The Sellalot Company gives a 25% discount (rounded to the nearest cent) on total purchases if the total purchases exceed $1000.00. **After** any applicable discount is applied, the Company adds on a shipping charge (rounded to the nearest cent) of 10% **of the purchase amount**, and adds on 8% sales tax (rounded to the nearest cent) **of the purchase amount** to get the final amount. By how many **cents** did Bill's final amount exceed Becky's final amount? Express your answer as an **exact whole number of cents**.

17. In the diagram (not necessarily drawn to scale), points B, C, D, and E are collinear. $\angle BAC \cong \angle CAD \cong \angle DAE$, $AB = 5.111$, $BC = 2.847$, and $CD = 4.015$. Find DE.



18. The black and white film a school uses comes in 100 foot bulk rolls. Each bulk roll costs $24.99. For each bulk roll, the school can get 19 exposure rolls of film out of the bulk roll. Find the cost for each of the 19 exposure rolls of film. Express your answer in standard **dollars** and **cents** notation, with the number of cents rounded to the nearest cent.

19. Let A represent a non-zero digit, and let B represent a non-zero digit. Find the sum of all distinct values of $(A+B)$ such that (A^5+B^5) is an integral multiple of $10A+B$. Express your answer as an **exact integer**. Do **not** use scientific notation.

20. Cindy, Jeffrey, and Lee (in that order) repeatedly take turns tossing an unfairly weighted cubical die. Assume the following probabilities for the uppermost face for any particular toss: $P(1)=0.1841$, $P(2)=0.1926$, $P(3)=0.2143$, $P(4)=0.1265$, $P(5)=0.1867$, and $P(6)=0.0958$. Cindy gets the first toss of the die with each person following in the order given. Find the probability that Jeffrey will be the first to toss a die that lands so that its uppermost face is an odd number.

2013 RA

Calculator Team

School ANSWERS

(Use full school name – no abbreviations)

______ Correct X 5 pts. ea. = [ ]

Note: All answers must be written legibly. Round answers to four significant digits and write in either standard or scientific notation unless otherwise specified in the question. Except where noted, angles are in radians. No units of measurement are required.

1. -1.036 OR -1.036×10^{0}
2. 35.96 OR 3.596×10 OR 3.596×10^{1}
3. 3106 (Must be this integer, $ optional.)
4. 301.5 OR 3.015×10^{2}
5. 12.22 OR 1.222×10 OR 1.222×10^{1}
6. 1.219 OR 1.219×10^{0}
7. 274.7 OR 2.747×10^{2} (Feet optional.)
8. 0.5921 OR $.5921$ OR 5.921×10^{-1}
9. 1.835×10^{14} (Must be in scientific notation.)
10. 171 (Must be this whole number, cents optional.)
11. 4.243 OR 4.243×10^{0}
12. 141.1 OR 1.411×10^{2}
13. 3023.165 (Must be this decimal.)
14. 4.345 OR 4.345×10^{0}
15. -4.155 OR -4.155×10^{0}
16. 137057 (Must be this integer, cents optional.)
17. 22.46 OR 2.246×10 2.246×10^{1}
18. 1.32 OR 1.32×10^{0} ($ optional.)
19. 33 (Must be this integer.)
20. 0.2614 OR $.2614$ 2.614×10^{-1}

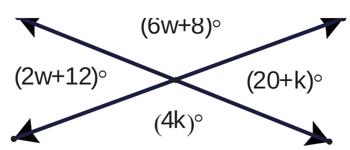
1. Let x be an integer such that $1 \le x \le 19$. Let k be the **number** of different values of x such that $(x-y)(x+y)=x^2-y^2$. Let w be the **number** of distinct positive integral factors of 2013. Find the product (kw).

2. The measures of the exterior angles of a triangle are in the ratio of $3:4:5$. The measures of the corresponding interior angles of this triangle are in the ratio of $a:b:c$ where a, b, and c are positive integers. Find the smallest possible value of $(2a+3b+4c)$.

3. Let a, b, and c be three distinct members of the set: $\{-5,-4,-3,-2,-1,0,1,2,3,4,5\}$ such that $a<b<c$. How many distinct ordered triples (a,b,c) exist such that $a+b+c=abc+1$?

4. Let $A=\{1,2,3,4\}$. Let k be the number of distinct subsets of A that contain the element 3. Let w be the number of distinct common tangents that two coplanar circles have if the two circles are externally tangent. Find the value of $(k+w)$.

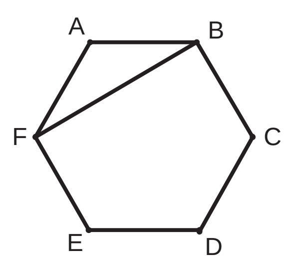
5. Let $y=mx+b$ be the equation of a line that contains $(2,7)$ and is perpendicular to the line whose equation is $x+3y=10$. In the diagram, two straight lines intersect with degree measures as shown. Find the value of $(m+b+k)$.



6. Find the value of $10111_{two}+2202_{three}+3212_{four}+544_{six}$. Express your answer in **base ten**.

7. Given the equation $\frac{8^{(3w)}}{4}=\left(\frac{1}{16}\right)^{(-2w-4)}$ and the circle whose equation is $x^2+14x+y^2+42y=879$. If r is the length of a radius of the circle, find the value of $(r+w)$.

8. In the diagram, $ABCDEF$ is a regular hexagon. Let $k=\frac{m\angle FAB}{m\angle BFA+m\angle ABF}$. Let w be the number of gallons of water that should be evaporated from 120 gallons of a solution that is 30% salt to obtain a solution that is 50% salt. Find the value of $(k+w)$.



9. Let $B = \{2x, 3x-4, 5x-32\}$. The arithmetic mean (average) of B is $(x+16)$. Let k be the largest of the three members of B. The length, width, and diagonal of a rectangle are integers. If the width were increased by 9, the area would be 120. Let A be the area of the original rectangle. Find the value of $(k+A)$.

10. Let k be the area of a triangle with side-lengths of 25, 34, and 39. After a $16\frac{2}{3}\%$ decrease in price, an item cost \$130. Let w be the number of dollars in the price of the item before the decrease. Find the value of $(k+w)$.

11. If $247_{thirteen}$ is changed to a base ten numeral, then k is the tens digit. One of the **exterior** angles of a rhombus has a degree measure of 120, and the length of the shorter diagonal of the rhombus is 7. Let p be the perimeter of the rhombus. Find the value of $(k+p)$.

12. The ratio of the volumes of two spheres is $729:15625$. If 9 is the length of the radius of the smaller sphere, let r be the length of the radius of the larger sphere. Let w be the smallest **positive** integer value of x such that $\left|18-|x-5|\right|>16.2$. Find the value of $(r+w)$.

13. The perimeter of a triangle is 176. The length of a radius of a circle which can be inscribed in the triangle is 15. Find the area of the triangle.

14. The equation of a circle is $3x^2+3y^2-6x+4y-1=0$. The area of this circle is $k\pi$. Find the value of k. Express your answer as an improper fraction reduced to lowest terms.

15. Mrs. Smith gave a test to 20 students. The first 12 students had an average score of 80, while the overall average score for the 20 students was 78. Find the average of the last 8 students.

2013 RA

School ANSWERS

Fr/So 2 Person Team

(Use full school name – no abbreviations)

Total Score (see below*) = ☐

NOTE: Questions 1-5 only are *NO CALCULATOR*

Note: All answers must be written legibly in simplest form, according to the specifications stated in the Contest Manual. Exact answers are to be given unless otherwise specified in the question. No units of measurement are required.

	Answer	**Score** (to be filled in by proctor)
1.	152	
2.	16	
3.	4	
4.	11	
5.	36	
6.	535 OR 535_{ten} OR 535_{10}	
7.	55	
8.	50	
9.	80	
10.	576	

TOTAL SCORE: ______

(*enter in box above)

Extra Questions:

11. 37

12. 29

13. 1320

14. $\frac{16}{9}$ (Must be this reduced improper fraction.)

15. 75

*** Scoring rules:**

Correct in 1st minute – 6 points

Correct in 2nd minute – 4 points

Correct in 3rd minute – 3 points

PLUS: 2 point bonus for being first In round with correct answer

1. Find the value of w if $\begin{vmatrix} 2w & 100 \\ 50 & 10 \end{vmatrix} = 2200$. Let k be the probability that the sum of the two top faces is six when randomly throwing two fair, standard cubical dice. Find the product (kw).

2. Let $A = \{1, 4, 5, 6, (2y+8), (y-4), (7y-4)\}$. The mode of A is a negative even integer. Let S be the sum of all possible distinct values for the mode of A. Find the value of S.

3. Find the smallest positive improper fraction that each of the following will divide exactly, resulting in integral quotients: $\frac{5}{13}$, $\frac{7}{26}$, and $\frac{11}{39}$. Express your answer as an improper fraction reduced to lowest terms.

4. Two sequences are defined as follows: $\begin{cases} a_1 = 2 \\ a_n = a_{(n-1)} + 3 \end{cases}$ and $\begin{cases} b_1 = 3 \\ b_n = b_{(n-1)} + 3.5 \end{cases}$. Find the arithmetic mean of the first 3 common terms of these sequences.

5. Let x be a radian measure such that $0 < x < \frac{\pi}{4}$. Find the smallest possible value of k such that $\csc(2x) + \cot(2x) = \cot(kx)$ for all possible values of x within the given conditions.

6. Let n be the smallest possible positive integer such that there are at least 3000 distinct committees possible when appointing a 3 member committee from n persons. Let r be the length of the radius of the inscribed circle of a triangle whose side-lengths are 25, 39, and 40. Find the value of $(n+r)$.

7. Let x be a two-digit positive integer and let y be a three-digit positive integer such that $\frac{3}{4} = \frac{x}{y}$. Let S be the sum of all distinct possibilities for the value of y. In a right triangle with all sides having integral lengths less than 15, let T be the value of 30 times the tangent of the largest possible acute angle. Find the value of $(S+T)$.

8. If $\log_3(2) = x$, $\log_3(20) = y$, then $2\log_9(135) = ax + by + c$. Let S be the sum of the lengths of the three dimensional vectors $(3, -6, 2)$ and $(-12, 4, 3)$. Find the value of $(a+b+c+S)$.

9. If x and y are positive integers, let k be the largest possible value for y such that $13x+17y=422$. A particle moves in a plane so that it is always equidistant from the point $(8,3)$ and the line whose equation is $y=5$. The equation of the path of the particle can be written as $y=ax^2+bx+c$. Find the value of $(abc+k)$.

10. Find the sum of all distinct values of x such that the three terms $x+7$, $5x-6$, and $7x+2$ taken in **some order** form an arithmetic sequence. Express your answer as an improper fraction reduced to lowest terms.

11. All lengths of the sides of a triangle are integers. One of the sides has a length of 12. The cosine of the smallest angle of the triangle is $\frac{3}{4}$. The length of one of the adjacent sides of the triangle for that angle is 12. Find the smallest perimeter for such a triangle.

12. Determine the positive difference between the number of digits in 93^{2013} and the units digit of 93^{2013}.

13. Let n be a positive integer such that $1 < n < 18$. For how many distinct values of n is $\left(\sum_{k=1}^{k=n}(k!)\right)$ the square of an integer?

14. Balls for the game of ICTM come only in boxes of 7 and 16. Thus, if you wanted 19 balls, you could not get 19 exactly with any combination of whole boxes of balls. Find the largest number of balls you could **not** get exactly with some combination of whole boxes of balls.

15. Let $i = \sqrt{-1}$. The quartic equation $x^4 - 2x^3 + kx^2 + wx + p = 0$ where k, w, and p are integers has -5 and $(6-9i)$ as two of its roots for x. Find the value of p

2013 RA

School ANSWERS

Jr/Sr 2 Person Team

(Use full school name – no abbreviations)

Total Score (see below*) = [ ]

NOTE: Questions 1-5 only are ***NO CALCULATOR***

Note: **All answers must be written legibly in simplest form, according to the specifications stated in the Contest Manual. Exact answers are to be given unless otherwise specified in the question. No units of measurement are required.**

Answer	Score (to be filled in by proctor)
1. 50	
2. –20	
3. $\frac{385}{13}$ (Must be this reduced improper fraction.)	
4. 38	
5. 1	
6. 37	
7. 1116	
8. 22	
9. 33	
10. $\frac{477}{40}$ (Must be this reduced improper fraction.)	

TOTAL SCORE: ________

(*enter in box above)

Extra Questions:

11. 28
12. 3960
13. 1
14. 89
15. 2925

*** Scoring rules:**

Correct in 1st minute – 6 points

Correct in 2nd minute – 4 points

Correct in 3rd minute – 3 points

PLUS: 2 point bonus for being first In round with correct answer

ORAL COMPETITION
ICTM REGIONAL 2013 DIVISION A

EXTEMPORANEOUS QUESTIONS

Give this sheet to the students at the beginning of the extemporaneous question period.

STUDENTS: You will have a maximum of 3 minutes TOTAL to solve and present your solution to these problems. Either or both the presenter and the oral assistant may present the solutions.

1. Give an example of a non-zero vector $\vec{v}$ for which the length of $\vec{v}$ is greater than the dot product of $\vec{v}$ with itself.

2. In two-space, two non-zero vectors which are both perpendicular to the same non-zero third vector are parallel to each other.
Explain why this is so and determine whether the same result holds in three-space.

ORAL COMPETITION
ICTM REGIONAL 2013 DIVISION A - SOLUTIONS AND JUDGES NOTES

1. Let $A=(5,4,-1)$, $B=(-23,-3,20)$ and $C=(1,3,2)$. Determine whether *A, B* and *C* are collinear.

SOLUTION

Form two vectors and see if one is a multiple of the other.
$\overrightarrow{AB}=-28\vec{i}-7\vec{j}+21\vec{k}$, $\overrightarrow{AC}=-4\vec{i}-\vec{j}+3\vec{k}$ and $\overrightarrow{BC}=24\vec{i}+6\vec{j}-18\vec{k}$
Note that $\overrightarrow{AB}=7\,\overrightarrow{AC}$, $\overrightarrow{BC}=-6\,\overrightarrow{AC}$ and $\overrightarrow{AB}=-\frac{7}{6}\,\overrightarrow{BC}$.
Any one of these three observations implies that the three points are collinear.

Alternately, $AB=7\sqrt{26}$, $BC=6\sqrt{26}$ and $AC=\sqrt{26}$, so $AB=AC+CB$.
Therefore the three points are collinear.

2. Let $P=(2,1,5)$, $Q=(-3,4,1)$ and $R=(3,0,t)$.
Find the value(s) of *t* for which ΔPQR is a right triangle.

SOLUTION

Form the vectors $\overrightarrow{PQ}=-5\vec{i}+3\vec{j}-4\vec{k}$, $\overrightarrow{PR}=\vec{i}-\vec{j}+(t-5)\vec{k}$ and $\overrightarrow{QR}=6\vec{i}-4\vec{j}+(t-1)\vec{k}$.
Two of these must be perpendicular, so their dot product must be 0.
From $\overrightarrow{PQ}\bullet\overrightarrow{PR}=0$ we get $-5-3-4(t-5)=0$, or $t = 3$. The right angle is at *P*.
From $\overrightarrow{PQ}\bullet\overrightarrow{QR}=0$ we get $-30-12-4(t-1)=0$, or $t = -\frac{19}{2}$. The right angle is at *Q*.
From $\overrightarrow{PR}\bullet\overrightarrow{QR}=0$ we get $6+4+(t-5)(t-1)=0$, or $t^2-6t+15=0$, which has no real solutions.
Thus there are two acceptable answers: $t = 3$ and $t = -\frac{19}{2}$.

Alternately, $PQ^2=50$, $QR^2=t^2-2t+53$ and $PR^2=t^2-10t+27$.
Solving $PQ^2+QR^2=PR^2$ gives -19/2. Solving $PQ^2+PR^2=QR^2$ gives $t = 3$.
There is no real solution to $PR^2+QR^2=PQ^2$. The two solutions are 3 and -19/2.

3. Let ΔABC be equilateral and let $\overrightarrow{AD} = \overrightarrow{AB} + \frac{3}{4}\overrightarrow{BC}$. Rounded to the nearest degree, find the measure of $\angle BAD$.

SOLUTION

It is convenient to make the side length 4 (or a multiple of 4).
The Law of Cosines gives $AD^2 = AB^2 + BD^2 - 2(AB)(BD)\cos(60°)$, or
$AD^2 = 4^2 + 3^2 - 2(4)(3)(1/2) = 13$, so $AD = \sqrt{13}$.
The Law of Sines in ΔBAD gives $\frac{\sin(\angle BAD)}{3} = \frac{\sin(60°)}{AD}$, or
$\sin(\angle BAD) = \left(\frac{3}{\sqrt{13}}\right)\left(\frac{\sqrt{3}}{2}\right) \approx .72058$. This makes $\angle BAD \approx 46°$.

An alternate solution could be:

$$\cos(\angle BAD) = \frac{\overrightarrow{AB} \bullet \overrightarrow{AD}}{\left|\overrightarrow{AB}\right|\left|\overrightarrow{AD}\right|} = \frac{\left|\overrightarrow{AB}\right|^2 + \frac{3}{4}\left|\overrightarrow{AB}\right|\left|\overrightarrow{BC}\right|\cos 120°}{\left|\overrightarrow{AB}\right|\left|\overrightarrow{AD}\right|}$$

$$\left|\overrightarrow{AD}\right|^2 = \left(\overrightarrow{AB} + \frac{3}{4}\overrightarrow{BC}\right) \bullet \left(\overrightarrow{AB} + \frac{3}{4}\overrightarrow{BC}\right)$$

$$= \left|\overrightarrow{AB}\right|^2 + \frac{3}{2}\left|\overrightarrow{AB}\right|\left|\overrightarrow{BC}\right|\cos 120° + \frac{9}{16}\left|\overrightarrow{BC}\right|^2$$

$$= \left|\overrightarrow{AB}\right|^2 \left(1 + \frac{3}{2}\left(-\frac{1}{2}\right) + \frac{9}{16}\right)$$

$$= \frac{13}{16}\left|\overrightarrow{AB}\right|^2$$

$$\left|\overrightarrow{AD}\right| = \frac{\sqrt{13}}{4}\left|\overrightarrow{AB}\right|$$

$$\cos(\angle BAD) = \frac{\left|\overrightarrow{AB}\right|^2\left(1 + \frac{3}{4}\left(-\frac{1}{2}\right)\right)}{\left|\overrightarrow{AB}\right|\left(\frac{\sqrt{13}}{4}\left|\overrightarrow{AB}\right|\right)} = \frac{\left|\overrightarrow{AB}\right|^2\left(\frac{5}{8}\right)}{\left|\overrightarrow{AB}\right|^2\left(\frac{\sqrt{13}}{4}\right)} = \frac{5}{2\sqrt{13}}$$

$$\angle BAD = \cos^{-1}\left(\frac{5}{2\sqrt{13}}\right) \approx 46°$$

ORAL COMPETITION
ICTM REGIONAL 2013 DIVISION A - SOLUTIONS AND JUDGES NOTES

EXTEMPORANEOUS QUESTIONS

1. Give an example of a non-zero vector $\vec{v}$ for which the length of $\vec{v}$ is greater than the dot product of $\vec{v}$ with itself.

SOLUTION

Since $\vec{v} \bullet \vec{v} = \left|\vec{v}\right|^2$ any non-zero vector whose length is less than 1 will work. Easy examples are $\frac{1}{2}\vec{i}$ and $\frac{2}{3}\vec{j}$. The respective lengths are $\frac{1}{2}$ and $\frac{2}{3}$ while the dot products are $\frac{1}{4}$ and $\frac{4}{9}$.

2. In two-space, two non-zero vectors which are both perpendicular to the same non-zero third vector are parallel to each other.
Explain why this is so and determine whether the same result holds in three-space.

SOLUTION

In a plane there is just one direction perpendicular to the direction of the third vector. Think vertical vs. horizontal or slope m vs. slope $-1/m$.
In three-space, however, there are infinitely many directions perpendicular to the direction of the third vector. Think of $\vec{i}$ and $\vec{j}$ for the first two and $\vec{k}$ for the third. Since $\vec{i}$ and $\vec{j}$ are not parallel the result does not hold in three-space.

Students may also reference simple geometry.